BSC 2011 Math review 2: Exponential equations

At the end of this exercise, students should be able to:

- Describe and solve equations for exponential growth and exponential decay
- Visualize how the rate of change varies across a graph showing an exponential relationship
- Explain how changing the value of the base in an exponential equation changes the slope of the curve

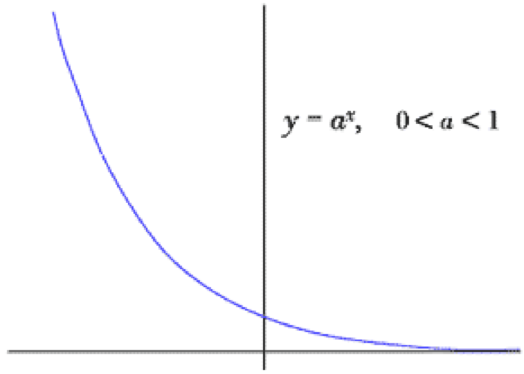
Before getting into exponential functions, let's first review the laws important laws of exponents:

1. $b^m . b^n \;=\; b^{m+n}$
2. $\frac{b^m}{bn} = b^{m-n}, b \neq 0$
3. $b^0 = 1$
4. $b^{-n} = \frac{1}{b^n}$
5. $(b^m)^n = b^{mn}$
6. $(ab)^n = a^n b^n$
7. $(\frac{a}{b})^n = \frac{a^n}{b^n}$
8. If $a^n = b$; $n = log(b)/log(a)$

Now let's discuss exponential functions.

The basic exponential function is defined as $f(x) = a^x$, where $a > 0, \quad \text{and} \quad a \neq 1$. Here *a* is called the base. If $0 < a < 1$, the function is decreasing (called exponential decay, see below in figure 1, case 1) and if $a > 1$, the function is increasing (called exponential growth, see below in figure 1, case 2).

Case 1: $0 < a < 1$, *Exponential Decay*



Case 2: $a > 1$, *Exponential Growth*

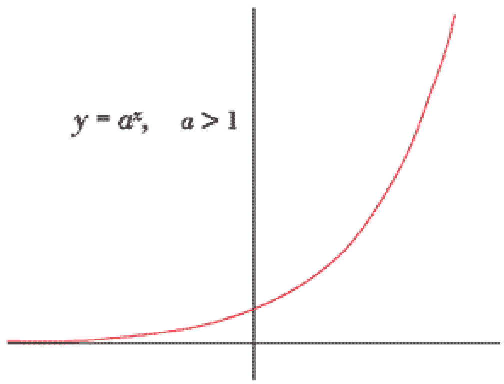


Figure 1. Exponential decay (case 1) and growth (case 2) functions.

In figure 2, we compare exponential functions with different bases for both base greater than 1 (exponential growth function) and between 0 and 1 (exponential decay).

Note how $f(x) = \frac{1}{10}^x$ represents exponential decay that is the mirror image of the exponential growth function shown by $f(x) = 10^x$. Note also in figure 2 how the value of *a* changes the slope of the curve. Note also that given the laws of exponents shown above, $f(x) = (\frac{1}{10})^x$= $f(x) = 10^{-x}$.

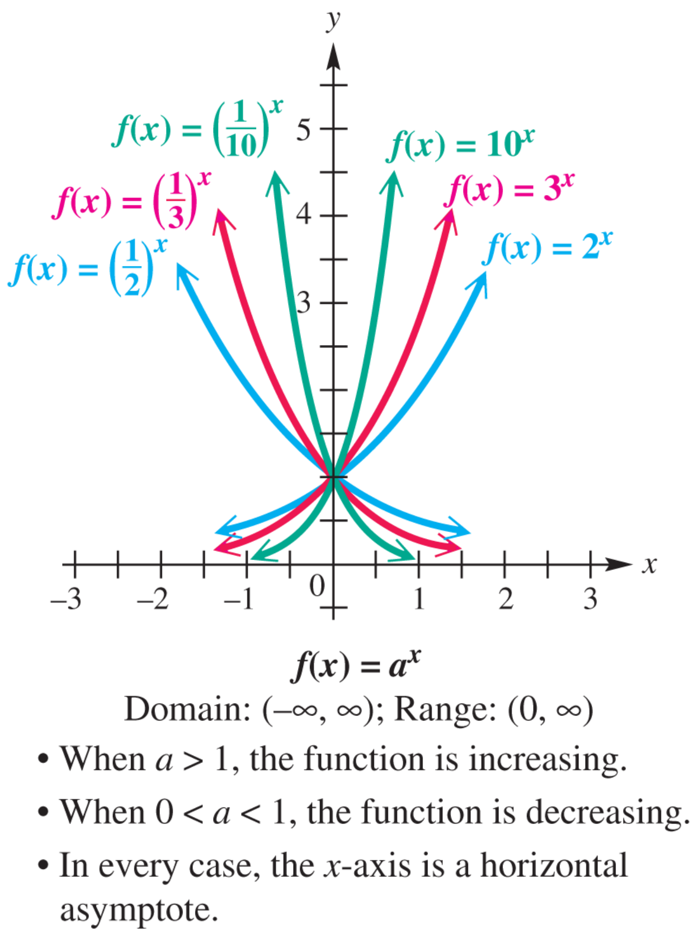


Figure 2. The equation $f(x) = a^x$ models exponential growth when $a > 1$, and exponential decay when a is a fraction.

<u>Exponential Growth and Decay Models:</u>

Exponential models are used to describe population growth and radioisotope decay, among other things. Graphs of these functions are drawn with time on the x – axis and quantity on the y – axis.

The general exponential growth/decay model is $A(t) = A_0\, a^t$. Here *A(t)* represents the quantity or number at any given time *t*. A_0 represents the initial quantity or initial number. *a* represents the multiplier for every time period.

In this course, we will use exponential growth functions to examine a certain type of population growth, called exponential growth, which is depicted in Figure 3. Each circle represents one individual (e.g. one bacterium). The vertical axis represents time. You can see from this diagram that with each passing generation, the number of individuals in the population doubles.

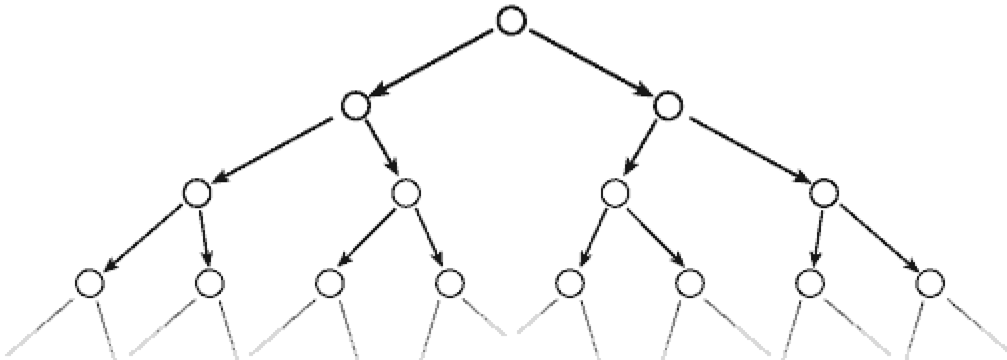


Figure 3. Population growth with a doubling of individuals with each generation (i.e. in the equation $A(t) = A_0 * (a)^t$, *a*=2, t=one generation.

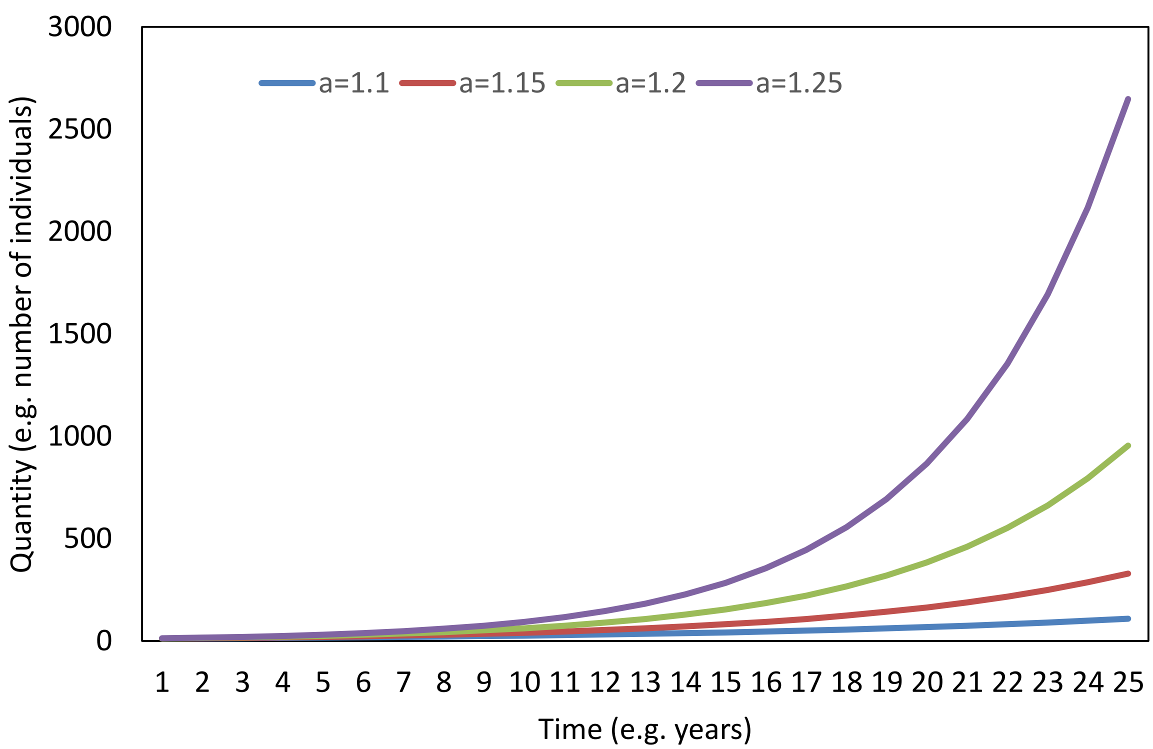


Figure 4. Exponential population growth over time which can modeled with the equation: $A(t) = A_0 * (a)^t$, for various values of a.

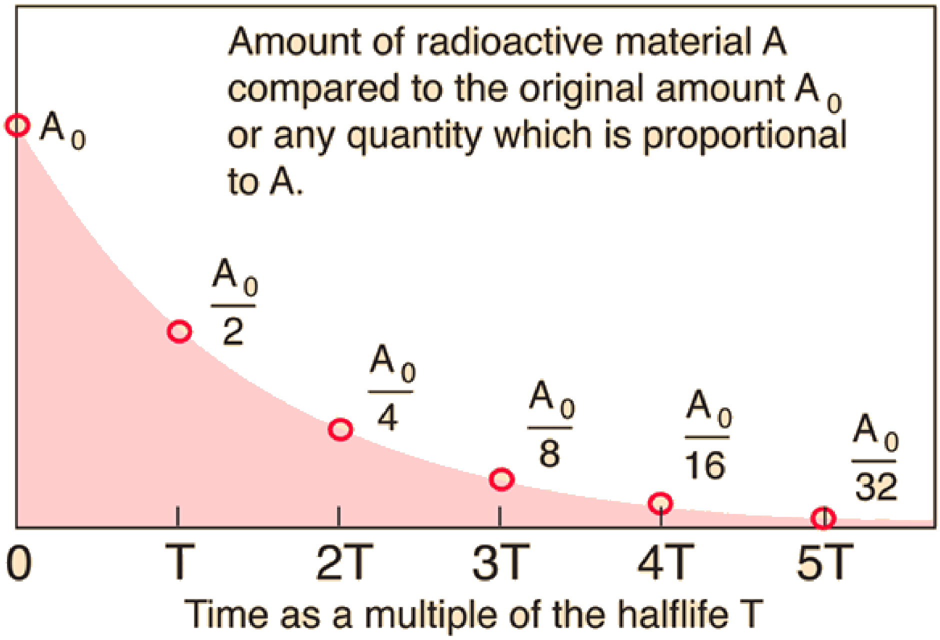


Figure 5. In exponential decay models, the amount (e.g. of radioactive material, or initial concentration of chemical) decreases by half with every half-life. This can modeled with the equations: $A(t) = A_0 * \left(\frac{1}{2}\right)^H$ $or\ A(t) = A_0 * (\frac{1}{2})^{\frac{t}{t_{half-life}}}$ or $A(t) = A_0 * (2)^{\frac{-t}{t_{half-life}}}$

Here, H is the number of half times that have elapsed. t is the time that has elapsed $t_{half-life}$ is the length of a half-life.

<u>Half – Life (Exponential decay)</u>

Half – life is defined as the time required for decaying half of the original quantity.

<u>Doubling – time (Exponential Growth)</u>

The time required to double the original quantity.

Examples of questions based on exponential functions:

These questions are similar to those you will see on quizzes and exams in BSC 2011. Below each question we provide a solution, both in written form and by using the formula. Note that you use whichever method you prefer, as long as you are able to get to the correct answer.)

1. The half-life of Carbon-14 is about 5700 years. You have found a fossil that you believe to be about 22,800 years old because it has _____ the normal (modern) expected ratio of Carbon-14
a. 1/4
b. 1/8
c. 1/16
d. not enough information given

Solution: Here we know since the half-life of Carbon – 14 is 5700, after 5700 years it becomes half of the original amount, in another 5700 years, that is a total of 11400 years, it becomes half of that amount, that is ¼ of the original amount. And in another 5700 years, that is a total of 17100 years, it becomes half of whatever left after 11400 years, and that is 1/8 of the original amount of Carbon – 14. In another 5700 years, that is a total of 22800 years, it becomes half of whatever left after 17100 years, and that is 1/16 of the original amount of Carbon – 14. So the correct answer is C. This is summarized in the table below:

Time (years)	**# of half-lives**	**Amount of C-14 (fraction of initial amount)**	**Amount of C-14 (% of initial amount)**
0	0	1	100
5700	1	1/2	50
11400	2	1/4	25
17100	3	1/8	12.5
22800	4	1/16	6.25
28500	5	1/32	3.125
34200	6	1/64	1.5625
39900	7	1/128	0.78125
45600	8	1/256	0.390625
51300	9	1/512	0.1953125

You can also calculate the amount of radioisotope left as function of time with the following equation:

$$A(t) = A_0 * (\frac{1}{2})^{\frac{t}{5700}}$$

Where A(t) is the quantify after time t has elapsed; N_0 is the initial quantity and t is time that has elapsed, in years (divided here by 5700 because the half-life of Carbon-14 is 5700 years). Since we want to know how much Carbon-14 is left as a proportion of the original amount (N_0) rather actual units, we can simplify this equation to:

$$A(t) = A_0 * (\frac{1}{2})^{\frac{t}{5700}} = A(t) = A_0 * (\frac{1}{2})^{\frac{22800}{5700}} = A_0 * (\frac{1}{2})^4 = A_0 * \frac{1}{16}$$

Again, using this equation we find that after 4 half-lives (or 22,800 years), the concentration of Carbon 14 is 1/16th of what it was when the fossil was created.

2. A test tube is inoculated with 100 cells of a bacterial strain that has a generation time of 30 minutes. The carrying capacity of the test tube for this strain is 6 billion cells. What will the bacterial population be after 90 minutes of culturing?

a. ~200 c. ~800

b. ~300 d. ~10,000

Solution: Here the bacterial strain has a generation time of 30 minutes (which is the doubling time, since bacteria reproduce by binary fission where one bacterium becomes two bacteria), so in 30 minutes 100 cells become 200 cells, in 60 minutes it will be 400 cells and in 90 minutes it will be 800 cells. So the correct answer is c. This is summarized in the table below:

Time (minutes)	**# generations**	**Number of bacteria**
0	1	100
30	2	200
60	3	400
90	4	800
120	5	1600
150	6	3200
180	7	6400
210	8	12800
240	9	25600
270	10	51200

This can also be modeled as an exponential growth problem, using the following equation:

$$A(t) = A_0 * 2^G \;\; OR \; A(t) = A_0 * 2^{\frac{t}{gt}} A(t) = A_0 * 2^{\frac{t}{30}}$$

Here, G is the number of elapsed generation. t is amount of elapsed time (e.g. minutes) and gt is the generation time (e.g. minutes per generation). This equation shows that every 30 minutes, the population will double (hence why the term a is 2). After 90 minutes, we find that the population is:

$$A(t) = 100 * 2^{\frac{90}{30}} = 100 * 2^3 = 100 * 8 = 800$$

3. A certain strain of bacteria that is growing on your kitchen counter doubles every 6 minutes. Assuming that you start with only one bacterium, how many bacteria could be present at the end of 126 minutes?

Solution: After 126 minutes, 21 generations have elapsed (126 minutes/6 minutes per generation). This means that the initial amount (1 bacterium) has doubled 21 times, hence is multiplied by 2^{21} as see in the following equation

$$A(t) = A_0 * 2^{\frac{t}{6}} = 1 * 2^{\frac{126}{6}} = 1 * 2^{21} = 2097152$$

Time (minutes)	# generations	Number of bacteria
0	0	1
6	1	2
12	2	4
18	3	8
24	4	16
30	5	32
36	6	64
42	7	128
48	8	256
54	9	512
60	10	1024
66	11	2048
72	12	4096
78	13	8192
84	14	16384
90	15	32768
96	16	65536
102	17	131072
108	18	262144
114	19	524288
120	20	1048576
126	21	2097152

4. If a person takes 125 milligrams of a drug at time 0, and the concentration of the drug left in the blood stream over time is represented by the function $A(t) = A_0\, 0.71^t$(where t is time in hours); what is the concentration of the drug in the bloodstream after 3 hours?

Solution: We know that for every hour, the concentration is multiplied by 0.71. Therefore, after one hour, there is 88.75mg left (125*0.71). After another hour (total 2 hours), there is ~63mg left (88.75*0.71). After another hour, for a total of 3 hours, there is ~44.74 mg of the drug in this person's bloodstream (63*0.71).

Time (hours)	Amount of drug (% of initial amount)	Amount of drug (mg)
0	100	125
1	71	88.75
2	50.41	63.01
3	35.79	44.74
4	25.41	31.76
5	18.04	22.55
6	12.81	16.01
7	9.10	11.37
8	6.46	8.07
9	4.58	5.73

This can also be solved with the following exponential equation:

$$A(t) = A_0\, 0.71^t = 125 * \ 0.71^3 = 44.74$$

5. There is 1000 tons of largemouth bass (a species of fish) in a lake in 2016. Your friend who works at the Florida Fish Wildlife Conservation Commission tells you that this population has been steadily growing, with a 10% increase in biomass every year. Assuming that this rate of population growth is maintained, what biomass of largemouth bass do you expect to find in that lake in 2019?

Solution: After one year (2017), there should be an additional 100 tons (1000tons*10%), for a total biomass of 1100tons. In 2018, there will be another 110tons (1100*10%), for a total of 1210 tons. In 2019, another 121 tons (1210*10%) will be added to the population for a total biomass of 1332 tons.

Time (years)	Year	Fish Biomass (tons)
0	2016	1000

1	2017	1100
2	2018	1210
3	2019	1331
4	2020	1464
5	2021	1611
6	2022	1772
7	2023	1949
8	2024	2144
9	2025	2358

This can also be solved with the following exponential equation, where t=time in years and 1.1 represents the yearly growth rate of the population (the amount that the population is multiplied by every year)

$$A(t) = A_0\, 1.1^t = 1000 * \ 1.1^3 = 1331$$

Bonus question (will NOT be asked to solve this type of question—where you have to solve for generation time—in quizzes and exams):

6. A single bacterium is introduced to a petri dish. After 260 minutes, there are now 8192 bacteria. How long is the generation time for this population of bacteria?

Solution:

$$8192 = 1 * 2^G$$

$$\log(8192) = \log(2^G)$$

$$\log(8192) = \log(2)G$$

$$\mathrm{G} = \frac{\log(8192)}{\log(2)} = \ 13$$

Thirteen generations have elapsed in 260 minutes. Therefore, the generation time is 20 minutes (260 minutes/13 generations).

If you need more refreshers on exponential functions and their interpretation, look at the following videos:

1. Video on exponential growth function from the Khan Academy: https://www.khanacademy.org/math/algebra2/exponential-growth-and-decay-alg-2/interpreting-the-rate-of-change-of-exponential-models/v/interpreting-change-in-exponential-models
2. Practice interpreting change in exponential growth models: https://www.khanacademy.org/math/algebra2/exponential-growth-and-decay-alg-2/interpreting-the-rate-of-change-of-exponential-models/e/modeling-with-exponential-functions
3. Video on exponential decay from the Khan Academy: https://www.khanacademy.org/math/algebra2/exponential-growth-and-decay-alg-2/interpreting-the-rate-of-change-of-exponential-models/v/interpreting-time-in-exponential-models
4. Practice interpreting change in exponential decay models: https://www.khanacademy.org/math/algebra2/exponential-growth-and-decay-alg-2/interpreting-the-rate-of-change-of-exponential-models/e/rewriting-and-interpreting-exponential-functions

5. Video showing word problems with exponential functions from the Khan Academy:https://www.khanacademy.org/math/algebra2/exponential-growth-and-decay-alg-2/intro-to-rate-of-exponential-growth-and-decay/v/word-problem-solving-exponential-growth-and-decay